Moving fronts for complex Ginzburg-Landau equation with Raman term

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Moving fronts, or optical shock-type solitons, are discussed for systems with gain and loss under the influence of the Raman effect. We present energy and momentum segment balance equations and establish the exact moving front solutions. We also show here that stationary and moving fronts also exist when we allow for various other nonlinear terms in the modified Ginzburg-Landau equation. [S1063-651X(98)09611-1]

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I. INTRODUCTION

The nonlinear Schrödinger equation (NSE) is a model equation describing a variety of short-pulse propagation phenomena in optics [1,2]. The NSE with nonconservative terms added is usually called the complex Ginzburg-Landau equation (CGLE). Particular areas of application of the CGLE are all-optical fiber transmission lines and passively mode-locked fiber and solid-state lasers. The NSE can be modified to include the influence of various physical phenomena on short-pulse generation and propagation. The behavior of ultra-short pulses changes under the influence of these terms. For example, third-order dispersion results in pulse asymmetry and radiation phenomena [2]. Fourth-order dispersion may result in solitons with oscillating tails [2]. Self-steepening causes the leading edge of a pulse to become more sharp [1]. There are various works on ''optical wave breaking'' as well. The inclusion of the Raman term results in a continuous downward shift in pulse frequency [3–6]. In the time domain, this represents the fact that the glass (optical fiber) response to the imposed field is not instantaneous. This delay of a few femtoseconds (fs) can affect propagation of fs signals. Gagnon and Belanger [7] showed that the exact form of the soliton self-frequency shift follows from a symmetry analysis of the equation.

Kink-type solutions for the NSE with Raman term present were discovered in [8]. In nonlinear optics, a kink is a shock wave which propagates undistorted in a dispersive nonlinear medium. Interestingly enough, when gain and loss terms are added to the NSE with the Raman term, the kink solution can still exist. In contrast to the NSE case, the front moves with a certain velocity which depends on the parameters of the equation.

In the present work we consider moving front solutions for the modified Ginzburg-Landau equation. We have developed a special technique to find the solution in analytic form which is based on energy and momentum balance equations [9]. In order to do this, we use the ansatz which follows from the point symmetry of the equation and use this ansatz in the energy and balance equations. Allowing for nonzero velocity generalizes this ansatz to cover the case of moving fronts. This technique allows us to find the coefficients of the solution in terms of the equation coefficients. We consider here two examples which allow us to obtain the solution this way: the CGLE with the Raman term and the CGLE with quintic terms. Other cases can also be considered and exact solutions can be found.

II. FUNDAMENTAL EQUATION

For a field $\psi(z,\tau)$ in a fiber with nonlinearity N, we may write the modified NSE in the following form:

$$i\psi_z+\frac{D}{2}\psi_{tt}+\psi N(|\psi|^2)=0. \tag{1}$$

Here D, the group delay dispersion coefficient, is positive for anomalous dispersion, and negative for normal dispersion. When we introduce the nonlinear and nonconservative terms, and consider the Raman and self-steepening effects, we obtain [10–13]

$$i\psi_z+\left(\frac{D}{2}-i\beta\right)\psi_{tt}+|\psi|^2\psi(1-i\epsilon) = i\delta\psi+\tau_R\psi(|\psi|^2)_t-is(|\psi|^2\psi)_t+(i\mu-\nu)\psi|\psi|^4. \tag{2}$$

Here the term with s, which is the self-steepening coefficient, and that with τ_R, which is the Raman coefficient, modify the complex Ginzburg-Landau equation [2]. In the CGLE, z is the normalized distance, t is the retarded time, ψ is the normalized envelope of the optical field, β describes the gain spectrum, δ is a constant gain (or loss if negative), and ϵ is a nonlinear gain (or two-photon absorption if negative). In general, some of these terms, which are additional to those in the nonlinear Schrödinger equation, are needed in the analysis of soliton pulse propagation in optical fibers [2,14]. We seek a suitable ansatz which can be used to solve some cases of the above equation. To do this, we first analyze the special case with $\beta=\epsilon=\mu=\delta=0$, and then use the same functional form in more general cases.

III. ANSATZ

Appropriate transformations for finding solutions for these types of equations can be formally described by considering the point-symmetry group of the equation. Each transformation then has an infinitesimal operator or generator associated with it [15,16]. Thus by setting $\psi=U(t)e^{-i\Omega z}$, and allowing for possible phase chirp by using U

$=F(t)\exp[i\phi(t)]$, with F and ϕ real, we obtain a complex equation. The real part is

$$\frac{D}{2}[F_{tt}-F\phi_t^2]+F(F^2+\Omega)=2\tau_R(F^2)F_t+sF^3\phi_t-\nu F^5, \tag{3}$$

while the imaginary part is

$$\frac{d}{dt}(F^2\phi_t)=-6DsF^3F_t=-\frac{3s}{2}D\frac{d}{dt}(F^4). \tag{4}$$

Equation (4) is nontrivial only if s is nonzero. In that case, we have

$$\phi_t=-\frac{3s}{2}D(F^2)+c/F^2,$$

where c is a constant of integration. Thus

$$\phi=-\frac{3s}{2}D\int F^2dt+c\int\frac{dt}{F^2}. \tag{5}$$

Then Eq. (3) transforms to

$$\frac{D}{2}F_{tt}+F^3-F(sc-\Omega)=2\tau_R(F^2)F_t-\bar{\nu}F^5+c^2\frac{D}{2}F^{-3}, \tag{6}$$

where

$$\bar{\nu}=\nu+\frac{3}{2}s^2D\left(1-\frac{3}{4}D^2\right). \tag{7}$$

We now let $F=\sqrt{w}$. This converts Eq. (6) to

$$\frac{D}{4}w_{tt}-\frac{Dw_t}{8w}+w^2-w(sc-\Omega)=\tau_R ww_t-\bar{\nu}w^3+c^2\frac{D}{2w}. \tag{8}$$

We look for solutions with $|\psi|$ approaching zero at one end (the low end), and approaching a nonzero constant at the other (high) end. If we set the constant c to zero, it is clear that Eq. (8) can be solved using $w=P\gamma[1+\tanh(\gamma t)]$, where P and γ are constants. This is true because

$$\frac{w_t}{w}=\gamma[1-\tanh(\gamma t)],$$

and then each term in Eq. (8) is of the form $\tanh^n(\gamma t)$ for $n=0$, 1, 2, or 3. We could equally well use the mirror image function $(t\to -t)$ for w.

Then

$$\phi=-\frac{3sD}{2}\int wdt=-\frac{3sDP}{2}[\gamma t+\log\cosh(\gamma t)].$$

Therefore the suitable ansatz for stationary front solutions is

$$\psi=\sqrt{P\gamma}\sqrt{1+\tanh(\gamma t)}e^{id\log\cosh(\gamma t)}e^{ikt-i\Omega z}, \tag{9}$$

where d and k are constants.

We are interested in moving front solutions. Hence, we can allow for solutions to move with velocity V by using

$$\gamma t\to\gamma(t-Vz)$$

in Eq. (9). In what follows, we prove that this form also gives the required solution. The real parameters of the solution (d, P, γ, k, V, and Ω) will be expressed in terms of the equation parameters (D, β, ϵ, δ, τ_R, μ, and ν). To relate the parameters, we will use an original technique of balance equations.

IV. SEGMENT ENERGY BALANCE FOR FRONT SOLUTIONS

First we consider the balance equations in general form with all the coefficients in Eq. (2) being nonzero. Let us convert to the moving group velocity frame by setting $\zeta=t-Vz$. Then, using the ansatz

$$\psi(\zeta,z)=f(\zeta)\exp[iz(KV-\Omega)],$$

we substitute it into Eq. (2). The resulting equation is

$$\frac{D}{2}f''-iVf'-(KV-\Omega)f+|f|^2f+is(|f|^2f)_\zeta$$
$$=i\delta f+i\beta f''+i\epsilon|f|^2f+(i\mu-\nu)|f|^4f+\tau_R f(|f|^2)_\zeta. \tag{10}$$

By multiplying by f^*, taking the complex conjugate, subtracting the two expressions, and integrating over ζ, we find that

$$\frac{D}{2}W(\zeta)-\frac{\beta}{2}(|f|^2)'+\frac{3}{4}s|f|^4-\frac{V}{2}|f|^2$$
$$=\delta\int|f|^2d\zeta+\epsilon\int|f|^4+\mu\int|f|^6d\zeta-\beta\int|f'(\zeta)|^2d\zeta, \tag{11}$$

where $W=\mathrm{Im}(f'f^*)$. This equation is a consequence of the energy balance. We may set $y=\gamma t$, and, for convenience, set the lower and upper limits at 0 and y, respectively, both for the evaluations on the left and the integrals on the right.

The Raman term only causes a frequency shift and not a gain or loss of energy, so it does not affect the segment energy balance equation for the front solutions. By using the above ansatz we can thus find the gain/loss contribution of each term. For a moving front solution the net gain must be zero. Following this procedure, the terms contain the independent functions y, $\tanh(y)$, $\tanh^2(y)$, and $\log[\cosh(y)]$. The overall coefficient of each term must therefore be zero. In this case the coefficients for the terms y and $\log[\cosh(y)]$ are the same:

$$\delta+2\epsilon P\gamma+4\mu P^2\gamma^2-\beta(k+d\gamma)^2=0. \tag{12}$$

Equating coefficients of $\tanh(y)$ we find

$$D(k+d\gamma)-V=2\beta(d^2\gamma+2kd-\tfrac{1}{4}\gamma)$$
$$-2\epsilon P-6\mu P^2\gamma-3sP\gamma, \tag{13}$$

while equating those for $\tanh^2(y)$ we obtain

$$4(Dd+\mu P^2)=\beta(4d^2-3)-6sP. \tag{14}$$

V. SEGMENT MOMENTUM BALANCE FOR FRONT SOLUTIONS

On the other hand, by multiplying Eq. (10) by $f^{*\prime}$, taking the complex conjugate, and adding the two expressions, we find that

$$\begin{aligned}-(kV-\Omega)(|f|^2)'+\frac{D}{2}(|f'|^2)'+\frac{1}{2}(|f|^4)'\\ =2\delta W-\frac{\nu}{3}\frac{d}{d\zeta}(|f|^6)-2Ws\frac{d}{d\zeta}(|f|^2)\\ +i\beta(f''f^{*\prime}-f^{*\prime\prime}f')+2\epsilon|f|^2W+2\mu|f|^4W\\ +\tau_R\left(\frac{d}{d\zeta}(|f|^2)\right)^2.\end{aligned} \tag{15}$$

By integrating with respect to ζ we find

$$\begin{aligned}(\Omega-kV)|f|^2+\frac{D}{2}|f'|^2+\frac{1}{2}|f|^4+\frac{\nu}{3}(|f|^6)\\ =2\int Wg(\zeta)d\zeta-2\beta\ \mathrm{Im}\int f''f^{*\prime}d\zeta\\ +\tau_R\int\left(\frac{d}{d\zeta}(|f|^2)\right)^2d\zeta,\end{aligned} \tag{16}$$

where

$$g(\zeta)=\delta+\epsilon|f|^2+\mu|f|^4-s\frac{d}{d\zeta}(|f|^2).$$

This equation is the result of the balance of momentum.

We now represent the complex function f using Eq. (9) in the form which allows for nonzero velocity. This leads to five momentum balance equations:

$$n_0+n_2+n_4=0, \tag{17}$$

$$n_1+n_3=0, \tag{18}$$

$$\begin{aligned}\gamma(\Omega-kV)+\frac{D}{2}\gamma\left(2kd\gamma+k^2-\frac{\gamma^2}{4}\right)+P\gamma^2\\ +\nu P^2\gamma^3+2(n_2+n_4)=0,\end{aligned} \tag{19}$$

$$\frac{D}{2}\gamma\left(d^2\gamma^2+2kd\gamma-\frac{\gamma^2}{4}\right)+\frac{1}{2}P\gamma^2+\nu P^2\gamma^3+n_3=0, \tag{20}$$

and

$$\left(d^2+\frac{1}{4}\right)\left(\frac{3D}{2}-2\beta d\right)+\nu P^2+2dP(\mu P+s)+P\tau_R=0, \tag{21}$$

where

$$n_0=ka-\beta\left[\frac{\gamma^2}{4}(2d\gamma+3k)+k^3\right]+\tau_R\frac{P}{2}\gamma^3,$$

$$n_1=kh_2+a(k+d\gamma)-\beta(k+3d\gamma)\left(\frac{\gamma^2}{4}+k^2\right),$$

$$n_2=kh_3+h_2(k+d\gamma)+ad\gamma-\beta h_4-\tau_RP\gamma^3,$$

$$n_3=h_3(k+d\gamma)+d\gamma h_2-\beta h_5,$$

and

$$n_4=d\gamma h_3-\beta d\gamma^3\left(d^2+\frac{1}{4}\right)+\tau_R\frac{P}{2}\gamma^3.$$

The coefficients used in the above equations are

$$a=\delta+P\gamma[\epsilon+\gamma(\mu P-s)],$$

$$h_2=P\gamma[\epsilon+2\gamma\mu P],$$

$$h_3=P\gamma^2[\mu P+s],$$

$$h_4=3\gamma\left[d\left(k^2-\frac{\gamma^2}{4}\right)+\gamma k\left(d^2-\frac{1}{4}\right)\right],$$

and

$$h_5=\gamma^2\left[d\gamma\left(d^2-\frac{3}{4}\right)+k\left(3d^2-\frac{1}{4}\right)\right].$$

These equations allow us to find the coefficients of the solution in terms of the coefficients of the fundamental equation. Henceforth, we set $s=0$, but of course, this restriction is not necessary when seeking other types of solutions.

VI. EXAMPLE 1: MOVING RAMAN FRONT SOLUTIONS

As an example, let us consider moving Raman front soliton solutions. Thus we set $\mu=\nu=s=0$, but take $\tau_R\neq 0$. Equation (2) then becomes

$$i\psi_z+\left(\frac{D}{2}-i\beta\right)\psi_{tt}+|\psi|^2\psi(1-i\epsilon)=i\delta\psi+\tau_R\psi(|\psi|^2)_t. \tag{22}$$

By using the solution of form (9), we can equate coefficients of the powers of $\tanh[\gamma(t-Vz)]$ and find the required constants. We obtain

$$d=\frac{D\pm\sqrt{D^2+3\beta^2}}{2\beta},$$

which is clearly real, and

$$P=(2\beta d-3D/8+d^2D/2)/\tau_R.$$

For convenience, we now define

$$r=P(1+2\epsilon d)/(1+4d^2)\quad\text{and}\quad a_1=3\beta/4+dD/2+\frac{D^2}{16\beta},$$

$$b_1 = r\left(2d - \frac{D}{2\beta}\right) - 2\epsilon P \quad \text{and} \quad c_1 = r^2/\beta - \delta.$$

Then

$$\gamma = [-b_1 \pm \sqrt{b_1^2 - 4a_1 c_1}]/(2a_1)$$

and

$$k = \left(r - \frac{\gamma D}{4}\right) \Big/ \beta.$$

The only restriction is that γ must be real, i.e., $b_1^2 > 4a_1c_1$. The velocity is

$$V = 2\epsilon P + k(D - 4\beta d) - \gamma(\beta + dD).$$

Finally, the frequency shift is

$$\Omega = \gamma^2\left(P\tau_R + \frac{D}{8} - \beta d\right) + k\left(\frac{Dk}{2} - \beta\gamma\right) - \gamma P.$$

These formulas can be used to present the relation between the equation and solution parameters in a simple way. From Eq. (9) we see that when t changes from $-1/2\gamma$ to $1/2\gamma$, the front intensity increases from 0.269 of its maximum to 0.731 of its maximum. We thus define the ''width'' of the front to be $1/|\gamma|$. In fact, at $t = -1/\gamma$, the intensity is 0.12 of its maximum, while at $t = 1/\gamma$, it is 0.88 of its maximum.

One example, giving the Raman kink width, height, and velocity versus ϵ for a given set of parameters, is shown in Fig. 1. Other dependencies of the soliton parameters versus parameters of the equation can also be presented on similar plots.

The stationary $(V=0)$ kink solution, obtained in [8], is a limit of this solution when $\beta, \epsilon \to 0$ and $D > 0$. Then most solution coefficients are zero, but

$$\gamma = \frac{-3}{2\tau_R} \quad \text{and} \quad \Omega = \frac{-9D}{8\tau_R^2}.$$

Thus the above solution reduces to

$$\psi = m\sqrt{D}\sqrt{1 - \tanh(2mt)}\,e^{2im^2Dz} = m\sqrt{D}e^{-mt}\sqrt{\text{sech}(2mt)}\,e^{2im^2Dz}, \tag{23}$$

where $m = -3/4\tau_R$, in agreement with [8].

VII. EXAMPLE 2: FRONT SOLUTION FOR QUINTIC EQUATION

We now consider the front solution for the quintic equation, i.e., we take $s = \tau_R = 0$ but $\mu, \nu \neq 0$. A Painleve analysis of the equation for this situation has been presented in [17]. We define $b_2 = (D\nu + 2\beta\mu)/(2\beta\nu - D\mu)$. Then

$$d = b_2 \pm \sqrt{b_2^2 + \frac{3}{4}}$$

and

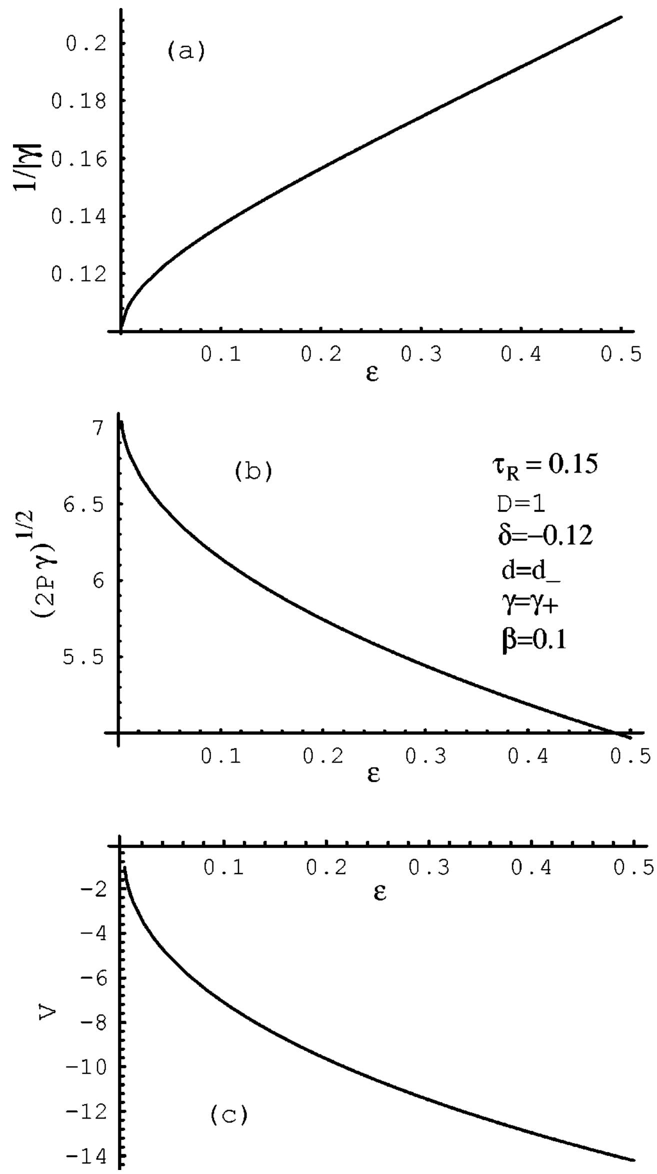

FIG. 1. (a) The width (as defined in text), (b) the height, and (c) the velocity of the kink solution of Eq. (22) as a function of ϵ. The parameters of the equation are given in the diagrams.

$$P = \sqrt{\left(\beta d^2 - dD - \frac{3}{4}\beta\right) \Big/ \mu}.$$

We now introduce

$$g = \frac{r_2}{4} + 2\beta d,$$

where

$$r_2 = 8P^2\frac{\nu + 2\mu d}{1 + 4d^2} - D,$$

$$a_2 = \frac{3}{4}\beta + dD - 3\mu P^2 + \frac{g r_2}{4\beta},$$

and

$$b_3=\frac{r}{\beta}\left(g+\frac{r_2}{4}\right)-2\epsilon P,$$

where r is as defined in the preceding section. Then

$$\gamma=\frac{1}{2a_2}\left[-b_3\pm\sqrt{b_3^2+4a_2\left(\delta-\frac{r^2}{\beta}\right)}\right]$$

and

$$k=\left(\frac{\gamma}{4}r_2+r\right)\bigg/\beta,$$

and the velocity is now

$$V=2\epsilon P+k(D-4\beta d)-\gamma(\beta+dD-4\mu P^2),$$

while the frequency offset is

$$\Omega=\gamma^2\left(\frac{D}{8}-\beta d-\nu P^2\right)-\gamma(P+\beta k)+\frac{D}{2}k^2. \qquad (24)$$

The actual solutions in these two sections satisfy both the energy [Eqs. (12)–(14)] and momentum [Eqs. (17)–(21)] balance equations.

$\beta=0$ solution family

As pointed out in [17], when $\beta=0$, there is a one-parameter family of solutions, as long as a consistency condition relating the equation parameters is satisfied. This solution has the same functional form as that above, with solution parameters simplifying to

$$b_2=-\nu/\mu$$

and

$$P^2=-dD/\mu.$$

Furthermore,

$$\gamma=\frac{4P\mu}{D}\frac{1+2d\epsilon}{8\nu d+\mu(1+20d^2)}. \qquad (25)$$

Now $V-kD$ can be written in terms of the equation parameters. The consistency condition is

$$\delta=\frac{3}{2}dD\gamma^2+\gamma P\left[\frac{c_2}{c_3}-\epsilon\right], \qquad (26)$$

where

$$c_2=\frac{5}{2}d-\epsilon\left(2\nu\frac{d}{\mu}+\frac{1}{4}\right),$$

and

$$c_3=5d^2+\left(2\nu\frac{d}{\mu}+\frac{1}{4}\right).$$

For example, we can specify D, ϵ, ν, and μ and then use Eq. (26) to find δ. Then, with V as an *arbitrary* velocity, we have

$$k=\left(V+2P\frac{c_2}{c_3}\right)\bigg/D,$$

with Ω given by Eq. (24) (with $\beta=0$). Thus moving front solutions exist even when the Raman term is absent.

VIII. CONCLUSION

We have found moving-front-type solitons of the extended CGLE. We have used the novel energy and momentum segment balance equations for analytical calculations. This method shows clearly the contribution of each term to the overall physical balance. In particular, the weighted losses and gains must add to zero for a valid front solution. The method allows us to find soliton solutions in a number of cases. In this paper we considered only particular cases of kinks for the equation with the Raman term and for the quintic CGLE. Other types of solutions (e.g., pulses) and solutions for the equation with other terms can be found in a similar fashion.

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